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ON OPTIMAL CONTROL PLANS FOR MULTIVARIATE QUANTITATIVE CHARACTERISTICS

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ABSTRACT

This article is devoted to the study of the asymptotic properties of the pair minimax design for a multivariate quantitative trait, asymptotic solutions for the parameters of the minimax and Bayesian designs that minimize the mean loss function.

Keywords: Asymptotics, function, minimax, multivariate quantity, statistic, sample, matrix, Bayesian design, object, nonlinear, covariance, product.

INTRODUCTION

In studies [3]–[4], the asymptotic properties of single-sample acceptance sampling plans (ASP) based on multivariate quantitative characteristics have been examined. Specifically, asymptotic solutions have been found for the parameters of minimax and Bayesian plans that minimize the average loss function.

The present article is devoted to studying the asymptotic properties of double-sampling minimax plans based on multivariate quantitative characteristics, in cases where the residual loss function is a nonlinear function concerning the proportion of defective items in the population.

Let each item in the given population be characterized by several features forming an S-dimensional vector $x=(x_1,x_2,...,x_S)$ with a normal distribution $\Phi(x_1,x_2,...,x_S)$ and a density function $\varphi(x_1,x_2,...,x_S)$.

Assuming that the mean vector $\mu=M(x)=(\mu_1,\mu_2,...,\mu_S)$ is unknown and the covariance matrix $\Sigma=M(x-\mu)(x-\mu)'$ is known.

An item is considered defective if $(x-t)'\Sigma^{-1}(x-t)>l$; otherwise, it is deemed acceptable.

The proportion of defective items in the population is given by

$$p=P\left\{(x-t)'\Sigma^{-1}(x-t)>l\right\}, \qquad (1)$$

where $t=(t_1,t_2,...,t_S)$ is a specified numerical vector and $l-$ is a given number.

It is known that the variable $X^2_{s,u}=(x-t)^1\Sigma^{-1}(x-t)$ follows a non-central chi-square distribution with S degrees of freedom and parameter $u^2=(\mu-t)^1\Sigma^{-1}(\mu-t)$.

Therefore

$$p=P(u)=1-H_S(l,u) \qquad (2)$$

where $H_S(l,u)$ is the distribution function of the variable $X^2_{s,u}$.

It can be shown that the function $H_S(l,u)$ is represented as:

$$H_S(l,u)=\int_o^l m_{s-1}(y)\left[\Phi(u+\sqrt{l+y})-\Phi(u-\sqrt{l+y})\right]dy \qquad (3)$$

where $\Phi(x)=\int_{-\omega}^{x}\varphi(z)dz,\ \varphi(z)=\frac{1}{\sqrt{2\pi}}\cdot e^{-z^2/2}$;

$m_0(x)=\delta(x)$ is the delta function;

$m_{S-1}(x)$ is the probability density function of the central chi-square distribution with S−1degrees of freedom, (S ≥2).

The statistical double-sampling plan consists of the following:

From a population of size N, a sample of size n_1 is taken. Let $x_i=(x_i^{(1)},x_i^{(2)},...x_i^{(s)}),\quad i=\overline{1,n}$ be the results of the inspection.

Let's construct the statistic $z_1^{\Sigma}=(\overline{x_1}-t)^1\Sigma^{-1}(\overline{x_1}-t)$, where *где* $\overline{x_1}=\frac{1}{n_1}\sum_{i=1}^{n_1}x_i$ and consider the following acceptance rule:

If $z_1^{\Sigma}\le k^2-\frac{\Delta}{\sqrt{n_1}}$, then the lot (i.e., the batch of items presented for inspection) is accepted, If $z_1^{\Sigma}\ge k^2+\frac{\Delta}{\sqrt{n_1}}$, then the lot is rejected, and If $\left|z_1^{\Sigma}-k^2\right|<\frac{\Delta}{\sqrt{n_1}}$, a second sample of size n_2 is taken, with inspection results denoted as $x_i=(x_i^{(1)},x_i^{(2)},...x_i^{(s)}),\ i=\overline{n_1+1,n_1+n_2}$.

After the second sampling, we construct the statistic $z_2^{\Sigma}=(\overline{x_2}-t)\Sigma^{-1}(\overline{x_2}-t)$, where *где* $\overline{x_2}=\frac{1}{n_2}\sum_{i=1}^{n_2}x_{n_{1+i}}$.

If $\frac{n_1\cdot z_1^{\Sigma}+n_2\cdot z_2^{\Sigma}}{n_1+n_2}\le k^2$, the lot is accepted; otherwise, it is rejected (rejected lots undergo 100% inspection).

The statistical plan is determined by selecting the control parameters n_1, n_2 and k.

The choice of a control plan can be made in various ways depending on specific requirements. One of these methods is selecting (n_1,n_2,k) based on minimizing the average losses from accepted and rejected populations presented for inspection.

Let $P(p,n_1,n_2,k)$ be the operating characteristic of the ASP (Acceptance Sampling Plan), i.e., the probability that a lot will be accepted, $R(p,n_1,n_2,k)$- be the residual function equal to the difference between the total average loss and the unavoidable loss associated with the given ASP. As noted in ASP studies [1-3], $R(p,n_1,n_2,k)$- has the form:

$$R(p,n_1,n_2,k)=\begin{cases}\left[(A-B)(p_0-p)\right]^{\upsilon}(1-P(p,n_1,n_2,k))+cM(n) & \text{при } p\le p_0\\ \left[(A-B)(p-p_0)\right]^{\upsilon}P(p,n_1,n_2,k))+cM(n) & \text{при } p\rangle p_0\end{cases}\qquad (3)$$

where $A=Na,\ B=N{\rm в},\ p_0=\frac{C}{a-в}$, a,в,c - are cost constants, and M(n) is the average sample size.

The ASP is considered optimal if the plan parameters (n1,n2,k) minimize the function $R(p,n_1,n_2,k)$. Depending on the availability of prior information about the quality of the lot (i.e., the proportion of defective items), there are two approaches to determining the optimal ASP: the minimax approach and the Bayesian approach.

Specifically, in the minimax approach, the optimal plan is determined by satisfying the following equalities:

$$R(n_{10,}n_{20,}k_0) = \sup_p R(n_{10,}n_{20,}k_{0,}p) = \inf_{n_1,n_2,k} \sup_p R(p,n_1 n_{2,}k)$$

The operating characteristic, i.e., the probability of lot acceptance, is given by

$$W_2(u) = P\left\{z_1^{\Sigma} \le k^2 - \frac{\Delta}{\sqrt{n_1}}\right\} + P\left\{\frac{n_1 \cdot z_1^{\Sigma} + n_2 \cdot z_2^{\Sigma}}{n_1 + n_2} \le k^2, \left|z_1^{\Sigma} - k^2\right| < \frac{\Delta}{\sqrt{n_1}}\right\} \qquad (4)$$

Clearly, the variable $n_j \cdot z_j^{\Sigma}$ ($j = 1,2,\ldots$) follows a non-central chi-square distribution with S degrees of freedom and non-centrality parameter $n_j \cdot u_0^S$. Therefore, for large n_j, the variable z_j^{Σ} is approximately normally distributed with parameters:

$$\left(u^2 + O(n_j^{-1}),\ \frac{2u}{\sqrt{n_j}} + O(n^{-\frac{2}{3}})\right)$$

Then, (4) takes the form:

$$W_2(u) = \Phi\left\{\frac{k^2 - \frac{\square}{\sqrt{n_1}} - u^2}{\frac{2u}{\sqrt{n_1}}}\right\} + \int_{k^2 - \frac{\Delta}{\sqrt{n_1}}}^{k^2 + \frac{\Delta}{\sqrt{n_1}}} \Phi\left\{\left[\frac{n_1 + n_2}{n_2} \cdot k^2 - \frac{n_1}{n_2} y\right] \frac{\sqrt{n_1}}{2u}\right\} d\Phi\left(\frac{y - u^2}{\frac{2u}{\sqrt{n_1}}}\right) + O(n^{-\frac{1}{2}})$$

Introducing the substitution $y = k^2 + 2w \cdot \frac{u}{\sqrt{n_1}}$

From here

$$W_2(u) \square \Phi\left\{-\frac{\Delta}{2u} + \frac{k^2 - u^2}{\frac{2u}{\sqrt{n_1}}}\right\} + \int_{-\frac{\Delta}{2u}}^{\frac{\Delta}{2u}} \Phi\left(-\frac{w}{\varepsilon} + \varepsilon \cdot \frac{k^2 - u^2}{\frac{2u}{\sqrt{n_1}}}\right) d\Phi\left(w + \frac{k^2 - u^2}{\frac{2u}{\sqrt{n_1}}}\right) \qquad (5)$$

where $\varepsilon = \sqrt{\frac{n_2}{n_1}}$.

Then, the average sample size is:

$$M(n) = n_1\left\{1 + \varepsilon\left[\Phi\left(\frac{\Delta}{2u} + \frac{k^2 - u^2}{\frac{2u}{\sqrt{n_1}}}\right) - \Phi\left(-\frac{\Delta}{2u} + \frac{k^2 - u^2}{\frac{2u}{\sqrt{n_1}}}\right)\right]\right\} \qquad (6)$$

Similarly to [1]–[3], the parameter k is determined from the equation:

$$W_j\big|_{p=p_0} = W_j(u_0, n, k) = \frac{1}{2}, \quad j = 1,2,\ldots \qquad (7)$$

It is easy to verify that the solution to (7) does not depend on j. From this, we obtain:

$$k = u_0 + O\left(n^{-\frac{1}{2}}\right).$$

Given this value of k, the relationships (4) - (6) take the form:

$$W_1(v) \square \Phi(v), \qquad (8)$$

$$W_2(v_1) \square \Phi\left(-\frac{\Delta}{2u_0}+v_1\right)+\int_{-\frac{\Delta}{2u}}^{\frac{\Delta}{2u}} \Phi\left(-\frac{w}{\varepsilon}+\varepsilon\cdot v_1\right) d\Phi\left(\varepsilon+v_1\right), \qquad (9)$$

$$M(n) \square n_1\left\{1+\varepsilon\left[\Phi\left(\frac{\Delta}{2u_0}+v_1\right)-\Phi\left(-\frac{\Delta}{2u_0}+v_1\right)\right]\right\}, \qquad (10)$$

where $v = \dfrac{u_0^{\,2}-u^2}{\dfrac{2u}{\sqrt{n}}}$, $v_1 = \dfrac{u_0^{\,2}-u^2}{\dfrac{2u}{\sqrt{n_1}}}$.

From the relationships (2) – (10), one can derive the asymptotic expansion of the residual loss function $R(n,k,p)$. Using this expansion, the following theorem regarding the parameters of the optimal double-sampling minimax acceptance sampling plan (ASP) can be proven:

Theorem: If Σ - is known, then for the minimax optimal plan, as $n_1 \to \infty$ u $n_2 \to \infty$ the following relationship holds:

$$n_{10}=n_{20}=\left[\frac{u_0}{2}\ \frac{1-W_2(n_0)}{W_1(n_0)}\right]^{\frac{2}{2+v}}\cdot\left(\frac{a-b}{c}Q_1\right)^{\frac{2}{2+v}}\cdot n^{\frac{2}{2+v}},$$

$$\min_{n_1}\max_{v_1} R(n_1,v_1)=\left[\frac{3}{2}W_1^{\frac{1}{2}}(n_0)n_0\cdot W_2(n_0)\right]^{\frac{2}{2+v}}\cdot\left(\frac{a-b}{c}Q\right)^{\frac{2}{2+v}}\cdot n^{\frac{2}{2+v}}$$

where $Q_1 = \dfrac{\partial}{\partial u}[1-H_s(\ell,u)]u=u_0,$.

$$W_1(u)=\Phi\left(\frac{\Delta}{2u_0}+\partial_1\right)+\Phi\left(\frac{\Delta}{2u_0}-v_1\right).$$

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