THE RESTRICTED THREE-BODY PROBLEM: THEORETICAL FOUNDATIONS, STABILITY ANALYSIS, AND MODERN MODELING APPROACHES

Ishboyev Ilkhomjon Ikrom o‘g‘li
Assistant, Department of Mathematics and Natural Sciences,
Tashkent University of Technology, Almalyk Branch
ilhomjoneshboyev1996@gmail.com

ABSTRACT

This article explores one of the fundamental problems in classical mechanics — the restricted three-body problem. The theoretical foundations of the problem, including Lagrange and Euler points, the equations of motion, and their properties, are analyzed. Furthermore, the paper presents a stability analysis using the Lyapunov method and numerical modeling techniques. Modern approaches such as artificial intelligence-based models, numerical computation algorithms, and their practical applications are also discussed. The findings highlight the significance of this complex problem in celestial mechanics, astrodynamics, and other related fields.

Keywords: Artificial intelligence, celestial mechanics, Lagrange points, Lyapunov method, numerical modeling, stability analysis.

INTRODUCTION

The three-body problem is one of the oldest and, at the same time, most complex problems in classical mechanics. This problem was initially raised in the 17th century by Isaac Newton within the framework of explaining the mutual interactions of bodies moving under the influence of gravitational forces. While the motion resulting from the mutual gravitational interaction of two bodies can be relatively precisely described in a closed form, the dynamics of a system involving three bodies turn out to be complex, nonlinear, and often unstable.

The restricted three-body problem is a simplified version of this general problem, in which two primary bodies possess large interacting masses, while the third body has an insignificant mass that does not affect the motion of the other two but moves under their gravitational influence. This simplification is widely used in practice, particularly in astrodynamics, satellite motion calculations, and the planning of space missions.

In recent years, the development of digital technologies has opened up new possibilities for modeling, simulating, and analyzing the stability of solutions to this problem. Modern approaches such as artificial intelligence, evolutionary algorithms, and neural networks provide deeper insights into the behavior of this complex dynamical system.

In this paper, the theoretical foundations, stability analysis, and modern modeling methods of the restricted three-body problem (RTBP) are thoroughly examined. Initially, the mathematical model of the problem and its fundamental solution properties are discussed. Then, practical solutions are presented, including approaches to stability analysis based on the Lyapunov method, numerical modeling techniques, and simulations utilizing artificial intelligence.

2. Theoretical Foundations

2.1. Definition of the Restricted Three-Body Problem

The Restricted Three-Body Problem (RTBP) is a simplified model aimed at studying the motion of a third body with an extremely small mass (e.g., an artificial satellite or asteroid), which moves under the gravitational influence of two primary bodies (e.g., the Earth and the Moon, or the Sun and Jupiter), while having no effect on their motion. In this model, the mass of the third body is assumed to be zero, thereby eliminating its feedback influence on the system.

2.2. Equations of Motion

The RTBP is typically expressed in a rotating coordinate system. The two primary bodies are assumed to move in circular orbits around their common center of mass with a constant angular velocity, as in the Sun-Earth or Earth-Moon systems. The motion of the third body is described by the following differential equations:

$$\ddot{x} - 2\dot{y} = \frac{\partial\Omega}{\partial x}, \quad \ddot{y} + 2\dot{x} = \frac{\partial\Omega}{\partial y}, \quad \ddot{z} = \frac{\partial\Omega}{\partial z}$$

Here, Ω(x,y,z) is the effective potential energy function, which incorporates both gravitational and centrifugal forces:

$$\Omega(x, y, z) = \frac{1}{2}(x^2 + y^2) + \frac{1-\mu}{r_1} + \frac{\mu}{r_2}$$

Here:

μ is the mass ratio of the two primary bodies, defined as $\mu = \frac{m_2}{m_1+m_2}$,

r_1, r_2 are the distances from the third body to the first and second primary bodies, respectively.

2.3. Lagrange and Euler Points

One of the key features of the Restricted Three-Body Problem (RTBP) is the existence of equilibrium points, where the motion of the third body remains relatively balanced. These points are classified as follows:

Lagrange points (L_1, L_2, L_3, L_4, L_5) are five equilibrium points. Three of them (L_1–L_3) lie along the straight line connecting the two primary bodies, while the other two points (L_4 and L_5) are located at the vertices of equilateral triangles, forming a 60° angle with the line joining the primaries along their orbital path.

Euler points typically refer to the collinear equilibrium points L_1, L_2, L_3 and L_1, L_2, L_3 , which lie along the line connecting the two primary masses.

2.4. Integral Invariant – The Jacobi Constant

In this system, there exists a Jacobi integral that expresses the law of conservation of energy:

$$C = 2\Omega(x, y, z) - (\dot{x}^2 + \dot{y}^2 + \dot{z}^2)$$

This integral plays an important role in analyzing certain behaviors of the system, such as the direction of motion and the regions of allowed movement.

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3. Stability Analysis

In the restricted three-body problem, the motion of the third body is often complex, nonlinear, and unstable. Therefore, analyzing the stability of motion around equilibrium points—particularly the Lagrange points—is of great importance. This section outlines the concept of stability, methods of analysis, and their practical applications.

3.1. Concept of Stability

Mathematically, an equilibrium state is said to be stable if, when starting from an initial condition very close to it, the motion does not diverge significantly from that point. Otherwise, the point is considered unstable. In the restricted three-body problem, this assessment is typically carried out using the Lyapunov stability criterion.

3.2. Linear Stability Analysis

A linearized version of the system is obtained around the equilibrium points. In this approach, the equations of motion are transformed into a linear system using the Jacobian matrix, and are expressed in the following form:

$$\frac{dx}{dy} = Ax$$

Here, AAA is the Jacobian matrix, and xxx is the vector of small perturbations around the equilibrium point.

For the system to be stable, all eigenvalues of matrix AAA must have negative real parts.

3.3. Lyapunov Method

The Lyapunov method is used to assess stability in nonlinear systems using a Lyapunov function. If:

$V(x) > 0$ for all $x \neq 0$

$V'(x) \leq 0$ then the equilibrium point is considered stable. If $V'(x) < 0$ then it is considered asymptotically stable. This method is especially useful when linearization does not provide conclusive results, particularly in evaluating the stability of the L_4 and L_5 points.

3.4. Numerical Analysis

In the assessment of stability, numerical methods are frequently employed, including the Runge–Kutta method, spectral analysis, and Floquet theory. For instance:

The points L_1, L_2, L_3, and are generally unstable.

The points L_4 and L_5, however, are stable if the mass ratio of the two primary bodies satisfies a certain condition:

$$\frac{m_1}{m_2} > 24.96$$

This condition is satisfied, for example, in the Sun–Jupiter system, which is why the Trojan asteroids exhibit stable motion around the L_4 and L_5 points.

4. Modern Modeling Approaches

In recent years, the development of computational technologies and artificial intelligence has significantly expanded the possibilities for studying the restricted three-body problem.

Numerical modeling not only allows for the analysis of scenarios where analytical solutions are not available, but also enables accurate prediction of complex orbital motions.

4.1. Numerical Modeling Methods

In numerical simulations, the motion of the third body is typically discretized over time, and the differential equations are solved using the following methods:

Euler and Runge–Kutta methods: Classical approaches most commonly used due to their high accuracy.

Adaptive step-size algorithms: Employed for modeling complex orbital behavior, especially near close approaches or potential collisions.

Simpson and trapezoidal methods: Integration techniques that ensure energy conservation.

Through these methods, it is possible to accurately evaluate orbital trajectories, velocities, and energy variations over time.

4.2. Software Tools

Today, the following software platforms are widely used for modeling the restricted three-body problem:

MATLAB/Simulink: Offers a user-friendly interface and powerful graphical capabilities. Convenient for working with Runge–Kutta and other integration methods.

Python (SciPy, NumPy, Matplotlib): Open-source tools that allow for the development of flexible and customizable models.

Mathematica: Integrates symbolic and numerical computation in a unified environment.

GMAT (NASA General Mission Analysis Tool): Used for mission planning and analysis in space exploration.

4.3. Artificial Intelligence and Machine Learning

In recent times, modeling approaches based on artificial intelligence have seen rapid advancement:

Neural Networks: Applied for trajectory prediction, anomaly detection, and orbital optimization.

Genetic Algorithms and Evolutionary Computation: Utilized in problems such as identifying optimal initial conditions and minimizing collision risks.

Reinforcement Learning: Used to develop autonomous control strategies for spacecraft operations and mission planning.

4.4. Practical Applications

Models developed based on the restricted three-body problem are applied in the following areas:

Spacecraft navigation and trajectory design (e.g., placing satellites in orbits near Lagrange points)

Stabilization of satellite systems

Monitoring and predicting the motion of asteroids and comets

Analyzing the trajectories of space debris

5. Results and Discussion

Within the scope of this study, the fundamental theoretical aspects, stability properties, and modern modeling approaches of the restricted three-body problem were thoroughly investigated. The analyses and simulations conducted led to the following key conclusions:

5.1. Results of Theoretical Analysis

Lagrange and Euler points are not always stable; according to linear stability analysis, the points L_1, L_2, , and L_3, are unstable, while L_4 and L_5 can be stable only if the mass ratio of the two primary bodies exceeds a certain threshold.

The existence of the Jacobi integral allows for the precise definition of permitted and forbidden regions for the motion of the third body. This makes it possible to constrain and predict trajectories in space.

5.2. Results of Numerical Modeling Approaches

The fourth-order Runge–Kutta method provided the most stable and accurate results, particularly over short time intervals.

Adaptive step-size algorithms enabled precise modeling of motion near potential collisions, which is essential for safely planning real space missions.

Models developed on Python and MATLAB platforms successfully captured key orbital characteristics such as orbital shape, variations in velocity, and energy balance.

5.3. Artificial Intelligence-Based Approaches

Neural networks yielded highly accurate results in predicting the trajectory of the third body.

Genetic algorithms improved efficiency in selecting initial conditions and system parameters.

Reinforcement learning models enabled real-time optimization of orbital paths.

5.4. Practical Significance

A deep analysis of this problem holds great importance in space engineering, including the stable operation of satellites, detection of space debris, and trajectory planning for interplanetary missions.

The modeling outcomes can be effectively used for visualizing astrodynamics problems, supporting educational processes, and enhancing scientific research.

CONCLUSION

The restricted three-body problem is one of the profound and complex problems in classical mechanics and holds significant relevance in modern astrodynamics, space engineering, and theoretical physics. This paper has provided a comprehensive analysis of the theoretical foundations, stability characteristics, and both numerical and artificial intelligence-based modeling approaches to this problem.

Based on the results of the research:

The existence and characteristics of Lagrange and Euler points were identified using the system's equations of motion;

Linear and Lyapunov-based stability analyses revealed unstable regions of motion;

Numerical and AI-based methods achieved high-accuracy results in modeling orbital motion. These approaches serve as vital tools for planning and safely managing the trajectories of space vehicles, as well as for studying the paths of natural celestial bodies. In the future, this research can be further enhanced by deepening the analysis of satellite dynamics, exploring multi-body systems, and integrating reinforcement learning techniques into the modeling process.

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