THE PROPERTIES OF TWO TYPES OF K-FUNCTIONS

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ANNOTATION

In this paper, two new types of functions are first proposed, and the related properties of these two types of functions are explored.

Keywords: K-function, iterative formula, indefinite integral, differential equations.

INTRODUCTION

In recent years, various special functions have been proposed and studied by many scholars; see [1]-[6] and the references therein. These research results are not only theoretical results, but also have the effect of solving engineering applications. This paper is aimed at two kinds of special functions, and explores their two properties of iterative formula and indefinite integral respectively. Throughout the paper, we define $Z^+ := \{0,1,2,3,\cdots\}$, $P_m^n := \dfrac{n!}{(n-m)!}$, and $0!:=1$.

PROBLEM FORMULATION AND MAIN RESULTS

Before presenting our main result, let us introduce the K-function.

Definition 1. The first type of K-function is defined by

$$K_1(x,n) := x^n \cos x, \text{ with } n \in Z^+. \tag{1}$$

Besides, the second type of K-function is defined by

$$K_2(x,n) := x^n \sin x, \text{ with } n \in Z^+. \tag{2}$$

For the graphs of the above two functions, see Figure 1 and Figure 2, respectively.

The iterative formulas for the K-functions are proposed as follows.

Theorem 1. For any $n \in Z^+$, one has

$$\begin{cases} \int K_1(x,n+2)dx = x^{n+2}\sin x + (n+2)x^{n+1}\cos x - (n+2)(n+1)\int K_1(x,n)dx; \\ \int K_2(x,n+2)dx = -x^{n+2}\cos + (n+2)x^{n+1}\sin x - (n+2)(n+1)\int K_2(x,n)dx. \end{cases}$$

Proof. Using integration by parts, it can be readily obtained that, for any $n \in Z^+$

$$\begin{cases} \int K_1(x,n+2)dx = x^{n+2}\sin x - (n+2)\int K_2(x,n+1)dx; \\ \int K_2(x,n+2)dx = -x^{n+2}\cos x + (n+2)\int K_1(x,n+1)dx. \end{cases}$$

It follows that

$$\int K_1(x,n+2)dx = x^{n+2}\sin x - (n+2)\left[-x^{n+1}\cos x + (n+1)\int K_1(x,n)dx\right]$$
$$= x^{n+2}\sin x + (n+2)x^{n+1}\cos x - (n+2)(n+1)\int K_1(x,n)dx;$$
$$\int K_2(x,n+2)dx = -x^{n+2}\cos + (n+2)\left[x^{n+1}\sin x - (n+1)\int K_2(x,n)dx\right]$$
$$= -x^{n+2}\cos + (n+2)x^{n+1}\sin x - (n+2)(n+1)\int K_2(x,n)dx.$$

This completes the proof.

Now we present another main result for the K-functions.

Theorem 2.

(i) $$\int K_1(x,4k)dx = \sum_{i=0}^{k} P_{4i}^{n} x^{4k-4i}\sin x + \sum_{i=0}^{k-1}\left(P_{4i+1}^{4k}x^{4k-4i-1}\cos x - P_{4i+2}^{4k}x^{4k-4i-2}\sin x - P_{4i+3}^{4k}x^{4k-4i-3}\cos x\right) + C, \quad \forall\, k\in Z^+,$$

(ii) $$\int K_1(x,4k+1)dx = \sum_{i=0}^{k}\left(P_{4i}^{4k+1}x^{4k-4i+1}\sin x + P_{4i+1}^{4k+1}x^{4k-4i}\cos x\right) - \sum_{i=0}^{k-1}\left(P_{4i+2}^{4k+1}x^{4k-4i-1}\sin x + P_{4i+3}^{4k+1}x^{4k-4i-2}\cos x\right) + C, \quad \forall\, k\in Z^+,$$

(iii) $$\int K_1(x,4k+2)dx = \sum_{i=0}^{k}\left(P_{4i}^{4k+2}x^{4k-4i+2}\sin x + P_{4i+1}^{4k+2}x^{4k-4i+1}\cos x - P_{4i+2}^{4k+2}x^{4k-4i}\sin x\right) - \sum_{i=0}^{k-1} P_{4i+3}^{4k+2}x^{4k-4i-1}\cos x + C, \quad \forall\, k\in Z^+,$$

(iv) $$\int K_1(x,4k+3)dx = \sum_{i=0}^{k}\left(P_{4i}^{4k+3}x^{4k-4i+3}\sin x + P_{4i+1}^{4k+3}x^{4k-4i+2}\cos x - P_{4i+2}^{4k+3}x^{4k-4i+1}\sin x - P_{4i+3}^{4k+3}x^{4k-4i}\cos x\right) + C, \quad \forall\, k\in Z^+,$$

(v) $$\int K_2(x,4k)dx = \sum_{i=0}^{k-1}\left(P_{4i+1}^{4k}x^{4k-4i-1}\sin x + P_{4i+2}^{4k}x^{4k-4i-2}\cos x - P_{4i+3}^{4k}x^{4k-4i-3}\sin x\right) - \sum_{i=0}^{k} P_{4i}^{4k}x^{4k-4i}\cos x + C, \quad \forall\, k\in Z^+,$$

(vi) $$\int K_2(x,4k+1)dx = \sum_{i=0}^{k}\left(-P_{4i}^{4k+1}x^{4k-4i+1}\cos x + P_{4i+1}^{4k+1}x^{4k-4i}\sin x\right) + \sum_{i=0}^{k-1}\left(P_{4i+2}^{4k+1}x^{4k-4i-1}\cos x - P_{4i+3}^{4k+1}x^{4k-4i-2}\sin x\right) + C, \quad \forall\, k\in Z^+,$$

(vii) $$\int K_2(x,4k+3)dx = \sum_{i=0}^{k}\left(-P_{4i}^{4k+2}x^{4k-4i+2}\cos x + P_{4i+1}^{4k+2}x^{4k-4i+1}\sin x + P_{4i+2}^{4k+2}x^{4k-4i}\cos x\right) - \sum_{i=0}^{k-1} P_{4i+3}^{4k+2}x^{4k-4i-1}\sin x + C, \quad \forall\, k\in Z^+,$$

(viii) $$\int K_2(x,4k+3)dx = \sum_{i=0}^{k}\left(-P_{4i}^{4k+3}x^{4k-4i+3}\cos x + P_{4i+1}^{4k+3}x^{4k-4i+2}\sin x + P_{4i+2}^{4k+3}x^{4k-4i+1}\cos x - P_{4i+3}^{4k+3}x^{4k-4i}\sin x\right) + C, \quad \forall\, k\in Z^+,$$

Proof. Using the iterative formula of Theorem 1, we can derive the following results respectively.

(i) $\int K_1(x,4k)dx$

$$\begin{aligned}
&= \left[P_0^{4k}x^{4k}\sin x + P_1^{4k}x^{4k-1}\cos x - P_2^{4k}x^{4k-2}\sin x - P_3^{4k}x^{4k-3}\cos x\right] \\
&\quad + \left[P_4^{4k}x^{4k-4}\sin x + P_5^{4k}x^{4k-5}\cos x - P_6^{4k}x^{4k-6}\sin x - P_7^{4k}x^{4k-7}\cos x\right] \\
&\quad + \cdots \\
&\quad + \left[P_{4k-4}^{4k}x^{4}\sin x + P_{4k-3}^{4k}x^{3}\cos x - P_{4k-2}^{4k}x^{2}\sin x - P_{4k-1}^{4k}x\cos x\right] \\
&\quad + P_{4k}^{4k}\sin x + C \\
&= \sum_{i=0}^{k} P_{4i}^{n}x^{4k-4i}\sin x + \sum_{i=0}^{k-1}\left(P_{4i+1}^{4k}x^{4k-4i-1}\cos x - P_{4i+2}^{4k}x^{4k-4i-2}\sin x\right. \\
&\quad \left. - P_{4i+3}^{4k}x^{4k-4i-3}\cos x\right) + C, \quad \forall\, k \in Z^{+}.
\end{aligned}$$

(ii) $\int K_1(x,4k+1)dx$

$$\begin{aligned}
&= \left[P_0^{4k+1}x^{4k+1}\sin x + P_1^{4k+1}x^{4k}\cos x - P_2^{4k+1}x^{4k-1}\sin x - P_3^{4k+1}x^{4k-2}\cos x\right] \\
&\quad + \left[P_4^{4k+1}x^{4k-3}\sin x + P_5^{4k+1}x^{4k-4}\cos x - P_6^{4k+1}x^{4k-5}\sin x - P_7^{4k+1}x^{4k-6}\cos x\right] \\
&\quad + \cdots \\
&\quad + \left[P_{4k-4}^{4k+1}x^{5}\sin x + P_{4k-3}^{4k+1}x^{4}\cos x - P_{4k-2}^{4k+1}x^{3}\sin x - P_{4k-1}^{4k+1}x^{2}\cos x\right] \\
&\quad + P_{4k}^{4k+1}x\sin x + P_{4k+1}^{4k+1}\cos x + C \\
&= \sum_{i=0}^{k}\left(P_{4i}^{4k+1}x^{4k-4i+1}\sin x + P_{4i+1}^{4k+1}x^{4k-4i}\cos x\right) \\
&\quad - \sum_{i=0}^{k-1}\left(P_{4i+2}^{4k+1}x^{4k-4i-1}\sin x + P_{4i+3}^{4k+1}x^{4k-4i-2}\cos x\right) + C, \quad \forall\, k \in Z^{+}.
\end{aligned}$$

(iii) $\int K_1(x,4k+2)dx$

$$\begin{aligned}
&= \left[P_0^{4k+2}x^{4k+2}\sin x + P_1^{4k+2}x^{4k+1}\cos x - P_2^{4k+2}x^{4k}\sin x - P_3^{4k+2}x^{4k-1}\cos x\right] \\
&\quad + \left[P_4^{4k+2}x^{4k-2}\sin x + P_5^{4k+2}x^{4k-3}\cos x - P_6^{4k-4}x^{4k}\sin x - P_7^{4k+2}x^{4k-5}\cos x\right] \\
&\quad + \cdots \\
&\quad + \left[P_{4k-4}^{4k+2}x^{6}\sin x + P_{4k-3}^{4k+2}x^{5}\cos x - P_{4k-2}^{4k+2}x^{4}\sin x - P_{4k-1}^{4k+2}x^{3}\cos x\right] \\
&\quad + P_{4k}^{4k+2}x^{2}\sin x + P_{4k+1}^{4k+2}x\cos x - P_{4k+2}^{4k+2}\sin x + C \\
&= \sum_{i=0}^{k}\left(P_{4i}^{4k+2}x^{4k-4i+2}\sin x + P_{4i+1}^{4k+2}x^{4k-4i+1}\cos x - P_{4i+2}^{4k+2}x^{4k-4i}\sin x\right) \\
&\quad - \sum_{i=0}^{k-1} P_{4i+3}^{4k+2}x^{4k-4i-1}\cos x + C, \quad \forall\, k \in Z^{+}.
\end{aligned}$$

(iv) $\int K_1(x,4k+3)dx$

$$= \left[P_0^{4k+3}x^{4k+3}\sin x + P_1^{4k+3}x^{4k+2}\cos x - P_2^{4k+3}x^{4k+1}\sin x - P_3^{4k+3}x^{4k}\cos x\right]$$
$$+ \left[P_4^{4k+3}x^{4k-1}\sin x + P_5^{4k+3}x^{4k-2}\cos x - P_6^{4k+3}x^{4k-3}\sin x - P_7^{4k+3}x^{4k-4}\cos x\right]$$
$$+\cdots$$
$$+ \left[P_{4k}^{4k+3}x^{3}\sin x + P_{4k+1}^{4k-2}x^{2}\cos x - P_{4k+2}^{4k-3}x\sin x - P_{4k+3}^{4k+3}\cos x\right] + C$$
$$= \sum_{i=0}^{k}\left(P_{4i}^{4k+3}x^{4k-4i+3}\sin x + P_{4i+1}^{4k+3}x^{4k-4i+2}\cos x - P_{4i+2}^{4k+3}x^{4k-4i+1}\sin x - P_{4i+3}^{4k+3}x^{4k-4i}\cos x\right)$$
$$+C.$$

(v) $\int K_2(x,4k)dx$

$$= \left[-P_0^{4k}x^{4k}\cos x + P_1^{4k}x^{4k-1}\sin x + P_2^{4k}x^{4k-2}\cos x - P_3^{4k}x^{4k-3}\sin x\right]$$
$$+ \left[-P_4^{4k}x^{4k-4}\cos x + P_5^{4k}x^{4k-5}\sin x + P_6^{4k}x^{4k-6}\cos x - P_7^{4k}x^{4k-7}\sin x\right]$$
$$+\cdots$$
$$+ \left[-P_{4k-4}^{4k}x^{4}\cos x + P_{4k-3}^{4k}x^{3}\sin x + P_{4k-2}^{4k}x^{2}\cos x - P_{4k-1}^{4k}x\sin x\right]$$
$$-P_{4k}^{4k}\cos x + C$$
$$= \sum_{i=0}^{k-1}\left(P_{4i+1}^{4k}x^{4k-4i-1}\sin x + P_{4i+2}^{4k}x^{4k-4i-2}\cos x - P_{4i+3}^{4k}x^{4k-4i-3}\sin x\right)$$
$$-\sum_{i=0}^{k}P_{4i}^{4k}x^{4k-4i}\cos x + C, \quad \forall\, k \in Z^{+}.$$

(vi) $\int K_2(x,4k+1)dx$

$$= \left[-P_0^{4k+1}x^{4k+1}\cos x + P_1^{4k+1}x^{4k}\sin x + P_2^{4k+1}x^{4k-1}\cos x - P_3^{4k+1}x^{4k-2}\sin x\right]$$
$$+ \left[-P_4^{4k+1}x^{4k-3}\cos x + P_5^{4k+1}x^{4k-4}\sin x + P_6^{4k+1}x^{4k-5}\cos x - P_7^{4k+1}x^{4k-6}\sin x\right]$$
$$+\cdots$$
$$+ \left[-P_{4k-4}^{4k+1}x^{5}\cos x + P_{4k-3}^{4k+1}x^{4}\sin x + P_{4k-2}^{4k+1}x^{3}\cos x - P_{4k-1}^{4k+1}x^{2}\sin x\right]$$
$$-P_{4k}^{4k+1}x\cos x + P_{4k+1}^{4k+1}\sin x + C$$
$$= \sum_{i=0}^{k}\left(-P_{4i}^{4k+1}x^{4k-4i+1}\cos x + P_{4i+1}^{4k+1}x^{4k-4i}\sin x\right)$$
$$+\sum_{i=0}^{k-1}\left(P_{4i+2}^{4k+1}x^{4k-4i-1}\cos x - P_{4i+3}^{4k+1}x^{4k-4i-2}\sin x\right) + C, \quad \forall\, k \in Z^{+}.$$

(vii) $\int K_2(x,4k+2)dx$

$$= \left[-P_0^{4k+2}x^{4k+2}\cos x + P_1^{4k+2}x^{4k+1}\sin x + P_2^{4k+2}x^{4k}\cos x - P_3^{4k+2}x^{4k-1}\sin x\right]$$
$$+ \left[-P_4^{4k+2}x^{4k-2}\cos x + P_5^{4k+2}x^{4k-3}\sin x + P_6^{4k+2}x^{4k-4}\cos x - P_7^{4k+2}x^{4k-5}\sin x\right]$$
$$+\cdots$$
$$+ \left[-P_{4k-4}^{4k+2}x^{6}\cos x + P_{4k-3}^{4k+2}x^{5}\sin x + P_{4k-2}^{4k+2}x^{4}\cos x - P_{4k-1}^{4k+2}x^{3}\sin x\right]$$
$$-P_{4k}^{4k+2}x^{2}\cos x + P_{4k+1}^{4k+2}x\sin x + P_{4k+2}^{4k+2}\cos x + C$$
$$= \sum_{i=0}^{k}\Big(-P_{4i}^{4k+2}x^{4k-4i+2}\cos x + P_{4i+1}^{4k+2}x^{4k-4i+1}\sin x$$
$$+P_{4i+2}^{4k+2}x^{4k-4i}\cos x\Big) - \sum_{i=0}^{k-1}P_{4i+3}^{4k+2}x^{4k-4i-1}\sin x + C, \quad \forall\, k \in Z^{+}.$$

(viii) $\int K_2(x,4k+3)dx$

$$= \left[-P_0^{4k+3}x^{4k+3}\cos x + P_1^{4k+3}x^{4k+2}\sin x + P_2^{4k+3}x^{4k+1}\cos x - P_3^{4k+3}x^{4k}\sin x\right]$$
$$+\left[-P_4^{4k+3}x^{4k-1}\cos x + P_5^{4k+3}x^{4k-2}\sin x + P_6^{4k+3}x^{4k-3}\cos x - P_7^{4k+3}x^{4k-4}\sin x\right]$$
$$+\cdots$$
$$+\left[-P_{4k}^{4k+3}x^{3}\cos x + P_{4k+1}^{4k+3}x^{2}\sin x + P_{4k+2}^{4k+3}x\cos x - P_{4k+3}^{4k+3}\sin x\right]$$
$$+C$$
$$= \sum_{i=0}^{k}\Big(-P_{4i}^{4k+3}x^{4k-4i+3}\cos x + P_{4i+1}^{4k+3}x^{4k-4i+2}\sin x$$
$$+ P_{4i+2}^{4k+3}x^{4k-4i+1}\cos x - P_{4i+3}^{4k+3}x^{4k-4i}\sin x\Big) + C, \quad \forall\, k \in Z^{+}.$$

This completes the proof.

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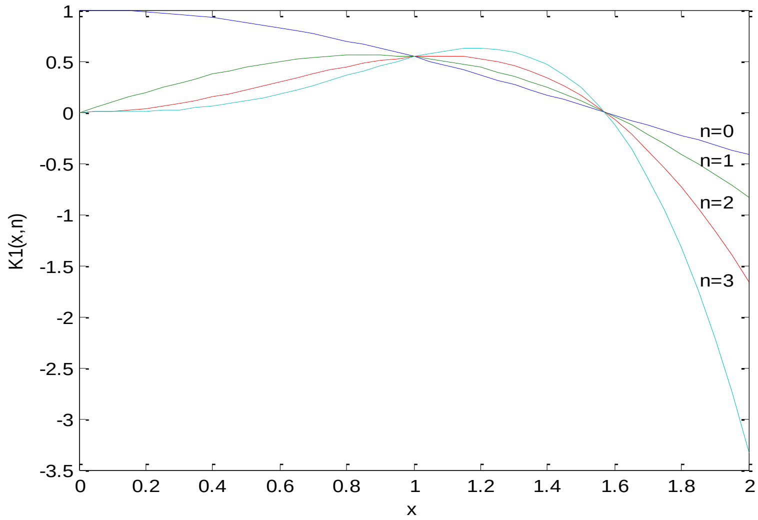

Figure 1. The first type of K-function.

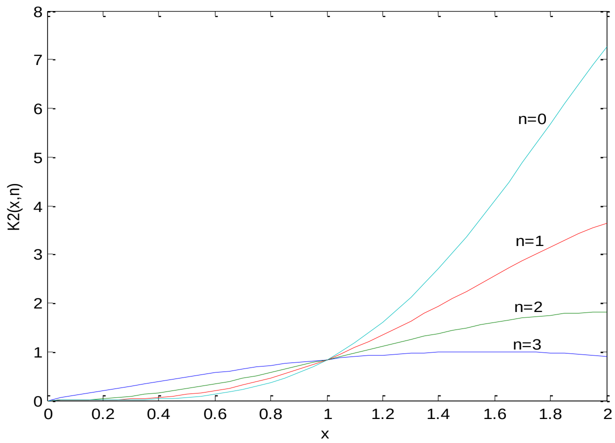

Figure 2. The second type of K-function.