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CONDITIONS FOR THE CONVERGENCE OF BRANCHING PROCESSES WITH IMMIGRATION STARTING FROM A LARGE NUMBER OF PARTICLES

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ANNOTATION

In this section, we study sufficient conditions for the convergence of a sequence of almost critical Galton -Watson branching processes with uniform immigration starting from a large number of particles.

Keywords ; Galton -Watson, branching processes , almost critical, random process, Skorokhod spaces

Let for everyone $n \in \mathbf{N}$ $\left\{\xi_{k,j}^{(n)}, k, j \in \mathbf{N}\right\}$ and $\left\{\varepsilon_k^{(n)}, k \in \mathbf{N}\right\}$ are two independent sets of independent, non-negative, integer- valued and identically distributed random variables. For each $n \in \mathbf{N}$, we define the process by the $\left\{X_k^{(n)},\ k \in \mathbf{N}_0\right\}$ following recursive relations

$$X_0^{(n)} = 0,\ X_k^{(n)} = \sum_{j=1}^{X_{k-1}^{(n)}} \xi_{k,j}^{(n)} + \varepsilon_k^{(n)},\quad k \in \mathbf{N}. \text{ (one)}$$

If we interpret the value $\xi_{k,j}^{(n)}$ as the number of descendants of the j -th particle in the -th $k-1$ generation , and the value $\varepsilon_k^{(n)}$ - as the number of particles immigrating into the population in the k -th generation, then the value $X_k^{(n)}$ is the number of particles in the population in the k -th generation. Due to this interpretation, process (1) is called the Galton -Watson branching process with immigration.

Let us assume that the values

$$m_n = E\xi_{1,1}^{(n)},\ \lambda_n = E\varepsilon_1^{(n)},\ \sigma_n^2 = D\xi_{1,1}^{(n)} \text{ and } b_n^2 = D\varepsilon_1^{(n)}$$

are finite for everyone $n \in \mathbf{N}$. Process (1) is called subcritical , critical and supercritical if $m_n < 1,\ m_n = 1,\ m_n > 1$, respectively. If $m_n \to 1$ for $n \to \infty$, then the sequence (1) is called almost critical.

Let , for each , be $n \in \mathbf{N}$ $\eta_0^{(n)}$ – a positive integer random variable, $\{\xi_{k,j}^{(n)}, k, j \in \mathbf{N}\}$ and $\{\varepsilon_k^{(n)}, k \in \mathbf{N}\}$ be two mutually independent collections of identically distributed, non-negative, integer- valued random variables that do not depend on $\eta_0^{(n)}$.

For each $n \in \mathbf{N}$ process $\left\{X_k^{(n)},\ k \in \mathbf{N}_0\right\}$, we define it as follows:

$$X_0^{(n)} = \eta_0^{(n)},\ X_k^{(n)} = \sum_{j=1}^{X_{k-1}^{(n)}} \xi_{k,j}^{(n)} + \varepsilon_k^{(n)},\ \ k \in \mathbf{N}.$$

assume the existence of second moments of the quantities $X_0^{(n)}$, $\xi_{k,j}^{(n)}$, $\varepsilon_k^{(n)}$ and keep the same notation for the means and variances of the quantities $\xi_{k,j}^{(n)}$ and $\varepsilon_k^{(n)}$ as at the beginning of §

Next, we define a step process $X_n(t)$ as an element of the Skorokhod space $D[0,T]$, setting $X_n(t) = X_{[nt]}^{(n)}, t \geq 0$, where [a] means the integer part of the number a. In what follows, the sign $\xrightarrow{D}$ will mean weak convergence in the Skorokhod topology (see [1]).

The following theorem gives an idea of the asymptotic behavior of the process $X_n(t), t \geq 0$ for $n \to \infty$ in the case when $X_0^{(n)} \xrightarrow{P} \infty$ (at the beginning there are "many" particles) and $m_n \to 1$ for $n \to \infty$ (almost the critical case).

Theorem . Let $m_n = 1 + \dfrac{\alpha}{d_n} + o\left(\dfrac{1}{d_n}\right)$, where $\alpha \in R$ and d_n be some sequence of positive numbers such that $\beta_n = nd_n^{-1} \to \beta < \infty$ for $n \to \infty$. Let $\gamma > 0$ the following conditions be satisfied for some: $n^{1-\gamma}\lambda_n \to \lambda$ and $n^{1-\gamma}b_n^2 \to b^2$, $n^{1-\gamma}\sigma_n^2 \to 0$, $n^{-\gamma}X_0^{(n)} \xrightarrow{P} X_0$ for $n \to \infty$, $\overline{\lim_{n\to\infty}}\, n^{-\gamma}EX_0^{(n)} < \infty$ and $EX_0 < \infty$. Then for any $T > 0$

$$\frac{X_n(t)}{n^\gamma} \xrightarrow{D} \eta(t),\ t \in [0,T] \text{ at } n \to \infty,$$

where the limiting random process $\eta(t)$ has the following form

$$\eta(t)=\begin{cases} X_0 e^{\alpha\beta t}+\dfrac{\lambda}{\alpha\beta}(e^{\alpha\beta t}-1), & \text{если} \quad \alpha\beta\neq 0, \\ X_0+\lambda t, & \text{если} \quad \alpha\beta=0. \end{cases}$$

Proof of the theorem. We represent the value $X_{k+1}^{(n)}$ in the form

$$X_{k+1}^{(n)}=X_k^{(n)}+(m_n-1)X_k^{(n)}+\lambda_n+M_k^{(n)}, \quad (2)$$

where $M_k^{(n)}=\sum_{j=1}^{X_k^{(n)}}(\xi_{k,j}^{(n)}-m_n)+\varepsilon_{k+1}^{(n)}-\lambda_n$. Obviously, this $M_k^{(n)},\ k\in\mathbf{N}_0$ forms a square-integrable martingale difference with respect to the flow of σ-algebras $F_k^{(n)}=\sigma\{X_0^{(n)},X_1^{(n)},\cdots,X_k^{(n)}\}$. Let $\eta_{nk}=\dfrac{X_k^{(n)}}{n^\gamma}$. Then from (2) we have

$$\eta_{n0}=\frac{X_0^{(n)}}{n^\gamma},\ \eta_{nk+1}=\eta_{nk}+(n(m_n-1)\eta_{nk}+\lambda_n n^{1-\gamma})\cdot\frac{1}{n}+\frac{1}{n^\gamma}M_k^{(n)} \quad (3)$$

It is easy to verify that

$$EX_k^{(n)}=m_n^k EX_0^{(n)}+\frac{m_n^k-1}{m_n-1}\lambda_n. \quad \text{(four)}$$

Further, applying Doob's inequality for martingales , we have

$$I_n=P\left(\sup_{0\le t\le T}\left|\frac{1}{n^\gamma}\sum_{k=0}^{[nt]}M_k^{(n)}\right|>\varepsilon\right)\le\varepsilon^{-2}n^{-2\gamma}E\left(\sum_{k=0}^{[nT]}M_k^{(n)}\right)^2=$$

$$=\varepsilon^{-2}n^{-2\gamma}\left(\sigma_n^2\sum_{k=0}^{[nT]}EX_k^{(n)}+[nT]b_n^2\right)$$

for any $\varepsilon>0$. From here and from (4), after simple calculations, we obtain the estimate

$$I_n \le \varepsilon^{-2} n^{-2\gamma}\left(\sigma_n^2 \frac{m_n^{[nT]+1}-1}{m_n-1} EX_0^{(n)} + \frac{\lambda_n \sigma_n^2}{m_n-1}\left(\frac{m_n^{[nT]+1}-1}{m_n-1}-[nT]\right)+[nT]b_n^2\right). \quad (5)$$

Obviously, if $\beta \ne 0$, then

$$m_n^n \to e^{\alpha\beta} \text{ at } n \to \infty.$$

Then from (5) we have

$$I_n \le \varepsilon^{-2}\alpha^{-1}(n^{1-\gamma}\sigma_n^2\beta_n^{-1}(e^{\alpha\beta T}-1)E\frac{X_0^{(n)}}{n^\gamma}+$$

$$+n^{1-\gamma}\lambda_n n^{1-\gamma}\sigma_n^2\beta_n^{-1}(\alpha\beta_n^{-1}(e^{\alpha\beta T}-1)-T)+$$

$$+\alpha n^{1-2\gamma}b_n^2 T)+o(1).$$

If $\beta = 0$, then

$$m_n^n = 1+\alpha\beta_n + o(\beta_n)$$

and from (5) it follows that

$$I_n \le \varepsilon^{-2}\left(n^{1-\gamma}\sigma_n^2 E\frac{X_0^{(n)}}{n^\gamma} + n^{1-\gamma}\lambda_n n^{1-\gamma}\sigma_n^2 + n^{1-2\gamma}b_n^2\right).$$

Then, taking into account the conditions of the theorem, we obtain that

$$I_n \to 0 \text{ at } n \to \infty.$$

Now, applying Theorem 3.1 from [2] and taking into account the last relations, we obtain

$$\max_{1\le k\le nT}\left|\eta_{nk} - Z_{nk}\right| \xrightarrow{P} 0 \text{ at } n\to\infty, \quad (6)$$

where the quantities Z_{nk} are determined by the recursive relations

$$Z_{n0}=\frac{X_0^{(n)}}{n^{\gamma}},\ Z_{nk+1}=Z_{nk}+(\alpha d_n^{-1}Z_{nk}+\lambda_n n^{1-\gamma})\cdot\frac{1}{n}.$$

Further, applying Theorem 3.2 from [2], we have

$$\sup_{0\le t\le T}\left|Z_n(t)-\eta(t)\right|=\max_{1\le k\le nT}\left|Z_{nk}-\eta\left(\frac{k}{n}\right)\right|\xrightarrow{P}0$$

for $n\to\infty$, where $Z_n(t)=Z_{n[nt]}$, the process $\eta(t)$ is a solution of the differential equation

$$d\eta(t)=(\lambda+\alpha\beta\eta(t))dt$$

with initial condition $\eta(0)=X_0$. From here and (6) it follows that

$$\sup_{0\le t\le T}\left|\frac{X_n(t)}{n^{\gamma}}-\eta(t)\right|\le\max_{1\le k\le nT}\left|\eta_{nk}-Z_{nk}\right|-\max_{1\le k\le nT}\left|Z_{nk}-\eta\left(\frac{k}{n}\right)\right|\xrightarrow{P}0$$

for $n\to\infty$, which was to be proved. The proof of the theorem is complete.

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